Working Paper No. 111

DIRECT REDUCTION OF LARGE CONCAVE PROGRAMS

by

ARTHUR M. GEOFFRION

December, 1966

WESTERN MANAGEMENT SCIENCE INSTITUTE

University of California, Los Angeles

University of California

Los Angeles

Western Management Science Institute

Working Paper No. 111

DIRECT REDUCTION OF LARGE CONCAVE PROGRAMS

by

A. M. Geoffrion

December, 1966

This working paper should be regarded as preliminary in nature, and subject to change before publication in the open literature. It should not be quoted without prior consent of the author. Comments are cordially invited.

This work was supported in part by contracts or grants from the Ford Foundation, National Science Foundation, and Office of Naval Research.

ABSTRACT

A very simple and flexible procedure is given for reducing a concave mathematical program to a finite sequence of subproblems, each involving a subset of the original constraints. While very much in the spirit of the "secondary constraint" idea, constraints are dropped as well as added from subproblem to subproblem. The reduction procedure is of quite general applicability, and is designed to be used with various concave programming algorithms for solving the subproblems. It seems particularly suited to facilitating the solution of problems with more constraints than could otherwise be handled. It is shown that an obvious specialization to linear programs yields the dual Simplex method, and that various "cutting-plane" methods can also be naturally described from the general viewpoint offered here.

DIRECT REDUCTION OF LARGE CONCAVE PROGRAMS

This paper discusses an extremely simple procedure for directly reducing a concave mathematical program to a finite sequence of subproblems, each with a subset of the original constraints. The procedure is quite flexible and general in its applicability. It is intended primarily for use with various concave programming algorithms to enable them to solve problems with more constraints than would otherwise be possible; hence how to solve the subproblems is deliberately left unspecified here. However, its use is by no means restricted to large problems.

Although conceived in the spirit of related reduction procedures developed by the author [3,4] for concave programs with some linear inequality constraints, it soon became evident that it can be viewed as a generalization of the "secondary constraints" procedure discussed by Dantzig [2] for linear programs; which, in turn, can be viewed as a way of implementing the dual Simplex method [8]. Indeed, we shall show that the procedure has a natural specialization for linear programs that corresponds exactly to the dual Simplex method. We further show that various "cutting-plane" methods can be viewed naturally as instances of the reduction procedure. Some of its aspects can also be found in a number of other works not cited here.

In section I we present the procedure, validate it, discuss its main arbitrary feature, and indicate its relation to an alternative class of reduction procedures. In section II we prove that it can be naturally

specialized to the dual Simplex method when applied to a linear program. The final section discusses difficulties that arise when there is an infinite number of constraints, or when some of the constraints are not explicitly available. Gomory's method for integer linear programming [5] and Kelley's method for concave programming [7] are shown to be instances of the reduction procedure for which these difficulties can be at least partially overcome.

I. THE REDUCTION PROCEDURE

Let $f, g_1, \ldots, g_m$ be concave functions defined on a convex set $X \subseteq R^n$, and let M be a finite index set (e.g., $M = \{1,2,\ldots,m\}$). The problem

(P) $\quad$ $\text{Maximize}_{x \in X}\ f(x)$ subject to $g_i(x) \geq 0, \quad i \in M$

will be reduced to a finite sequence of subproblems of the form

(P_S) $\quad$ $\text{Maximize}_{x \in X}\ f(x)$ subject to $g_i(x) \geq 0, \quad i \in S \subseteq M.$

Assume that a subset S^o is known such that (P_{S^o}) admits an optimal solution x^{S^o} (with $f(x^{S^o}) < \infty$), and assume further that (P_S) admits an optimal solution whenever it admits a feasible solution and its maximand is bounded above on the feasible region by $f(x^{S^o})$.

Under the above assumptions, we shall show that the following procedure is well-defined and terminates in a finite number of steps.

The Reduction Procedure

Step 0: Put $\bar{f} = \infty$ and $S = S^o$, where S^o is any subset of M such (P_{S^o}) admits a finite optimal solution.

Step 1: Solve (P_S) for an optimal solution x^S if one exists; if none exists (i.e., (P_S) is infeasible), then terminate with the message "(P) infeasible". Define $V_S \triangleq \{i \in M\text{-}S:\ g_i(x^S) < 0\}$. If $V_S = \emptyset$, terminate with the message "x^S is an optimal solution of (P)"; otherwise, go on to Step 2.

Step 2: Put v equal to any subset of M such that $\{v \cap V_S\} \neq \phi$. If $f(x^S) < \bar{f}$, replace S by $E_S \cup v$, where $E_S = \{i \in S: g_i(x^S) = 0\}$, and $\bar{f}$ by $f(x^S)$; otherwise (i.e., if $f(x^S) = \bar{f}$), replace S by $S \cup v$. Return to Step 1.

This procedure simply goes from one subproblem to the next by adding at least one constraint that is violated at an optimal solution of the current subproblem, while deleting the amply satisfied constraints so long as the value of the objective function is decreasing. Eventually a subproblem is encountered that is either infeasible, in which case (P) obviously must be infeasible, or has an optimal solution that is also feasible in (P), in which case that solution obviously must solve (P).

To show that the subproblems which arise are either infeasible or admit an optimal solution, in view of our assumptions it is enough to show inductively that the sequence $\langle f^S \rangle$ is non-increasing, where f^S is the supremum of the maximand of (P_S) (let $f^S = -\infty$ if (P_S) is infeasible). Certainly $f^{S \cup v} \leq f^S$, and $f^{E_S \cup v} \leq f^{E_S}$. We assert that $f^{E_S} = f^S$, which yields the desired monotonicity of $\langle f^S \rangle$. The assertion is an easy consequence of

Lemma 1.1: Let x^S be optimal for (P_S). Then $g_j(x^S) > 0$ and $j \in S$ implies that x^S is also optimal for (P_{S-j}).

Proof: Certainly $f^{S-j} \geq f(x^S)$. Suppose that $f^{S-j} > f(x^S)$. Then there exists a point x' feasible in (P_{S-j}) such that $f(x') > f(x^S)$. We may assume $g_j(x') < 0$, or else x' would contradict the optimality of x^S in

(P_S). By the concavity of f and the g_i, $i \in S$, and the convexity of X, it follows that for λ positive but sufficiently small the point $\lambda x' + (1-\lambda)x^S$ is feasible in (P_S). But then $f(\lambda x'+(1-\lambda)x^S) \geq \lambda f(x') + (1-\lambda)\, f(x^S) > f(x^S)$, which contradicts the definition of x^S. Hence $f^{S-j} = f(x^S)$, and x^S must be optimal for (P_{S-j}).

Thus far we have shown that the procedure is well-defined, and that the sequence $\langle f(x^S)\rangle$ is non-increasing. Since Step 2 only deletes amply satisfied constraints from S (before adding v) when the maximand has just strictly decreased, it follows from the finiteness of the number of possible subsets of M that $\langle f(x^S)\rangle$ can remain constant for only a finite number of consecutive iterations. Again appealing to the finiteness of the number of possible trial sets, we see that finite termination is established.

Theorem 1.1: The reduction procedure terminates in a finite number of steps with either (a) an optimal solution of (P), or (b) the identification of a subset of the constraints of (P) that are collectively infeasible over X. Moreover, in case (a) a monotone decreasing sequence $\langle f(x^S)\rangle$ of upper bounds on the optimal value of (P) is obtained.

Choice of v

There is considerable latitude in the choice of v at Step 2. The only requirement is that it must include the index of at least one constraint violated by the current x^S. As will be discussed more fully in section III, there are cases in which it is difficult or expensive to identify any

violated constraint, let alone all of V_S. But when V_S is readily available, propitious choice of v can heighten computational efficiency.

There are at least two conflicting considerations in the choice of v, depending on whether one focuses on making the subproblems easy to solve or on making the number that must be solved small. On one hand, the fewer the indices of which v is composed the easier it will probably be to solve the next subproblem starting from the current x^S. When a "primal feasible" optimization method is used to solve the subproblems, for example, it can be applied starting with the last x^S to each g_j, $j \in v$, in turn, until a feasible solution is found for the new subproblem (or until it is determined that the new subproblem is infeasible). A reasonable choice of v from this point of view might be to let it be a singleton (the index of the most violated constraint, say). On the other hand, it would seem that by taking v large (all of V_S or even more) the number of subproblems to be solved will be relatively small. This point of view is suggested by interpreting the reduction procedure as minimizing $f(x^S)$ over $S \subseteq M$; the more violated constraints are imposed at each iteration, the greater will be the decrease of $f(x^S)$ at next subproblem. The compromise between these two points of view which tends to minimize total computation time is probably quite problem-dependent.

Relation to Another Reduction Procedure

When the g_i are all linear, or when any nonlinear constraints have been incorporated into X, we have shown elsewhere [3,4] how to reduce (P) to a sequence of subproblems of the form

(P_S') $\qquad$ $\text{Maximize}_{x \in X}\ f(x)$ subject to $g_i(x) = 0,\ i \in S \subseteq M.$

The advantage of such subproblems is clear: linear equality constraints are often easier to deal with than the corresponding inequality constraints (especially when some of these constraints are simple non-negativity requirements). Without going into details, we can indicate that this subproblem simplicity is purchased at the price of a slightly more complicated reduction strategy and proof of termination.

One variant [4] chooses the next subproblem from the current one by the addition or deletion of a single constraint according to random selection from a certain set of candidates. It is interesting to note that Markov chain analysis and computational results suggest that termination then takes place on the average in something less than 2N subproblems, where N is the number of mistakes made in identifying (by the choice of the initial (P'_S)) the truly restrictive constraints of (P). Other variants coincide, when (P) is a linear program, with the primal or dual Simplex methods (cf. the next section).

Unfortunately subproblems of the form (P'_S) are not helpful when some of the g_i are nonlinear, for then the feasible region need no longer be convex.

II. RELATION TO THE DUAL SIMPLEX METHOD

The fact that a feasible solution of (P) is not obtained until the final step, and that $\langle f(x^S)\rangle$ is monotone decreasing to the optimal value of (P), suggests the adjective "dual" in describing the reduction procedure. Indeed, in this section we shall show that the procedure can be specialized in a natural way to yield Lemke's dual Simplex method [8] when (P) is a linear program.

Let $\quad f(x) = cx,$

$g_i(x) = x_i$, $M = \{1,\ldots,n\}$, and $X = \{x: Ax = b\}$ hold for (P), where c is $1 \times n$, b is $m_1 \times 1$, and A is $m_1 \times n$. The dual Simplex method is initiated with some set B^o of variables designated as "basic" which yields, from the "reduced costs" of an associated tabular representation of (P) (see below), a feasible solution of the dual to (P). Assuming that the successive feasible solutions to the dual are non-degenerate, we shall prove

<u>Theorem 2.1</u>: If S^o is taken as $M-B^o$ and ν always as the most violated constraint, then the set of non-basic variables at the ν^{th} iteration of the dual Simplex method coincides with E_S at the ν^{th} iteration of the reduction procedure, and the ν^{th} basic solution coincides with the ν^{th} x^S.

It is necessary to give a brief rendering of the dual Simplex method in order to establish the notation used in the proof. More complete details may be found, for example, in [6] or [8].

Problem (P) can be restated as one of maximizing z subject to $x \geqq 0$ and the following equality constraints stated as a tableau (m_1+1 by $n+2$) of detached coefficients:

z	x	= 1
1	$-c$	0
0	A	b

At any given iteration there is specified a collection B of m_1 basic variables such that A_B^{-1} exists, where A_B is formed by extracting columns from A according to B, and such that $\bar{c} \equiv (c_B A_B^{-1})A - c \geqq 0$, where c_B is similarly formed by extraction according to B. Moreover, the equality constraints are re-expressed as:

z	x	= 1
1	$(c_B A_B^{-1})A - c$	$c_B A_B^{-1} b$
0	$A_B^{-1} A$	$A_B^{-1} b$

If $\bar{b} = A_B^{-1} b \geqq 0$, then it is easily shown that an optimal solution of (P) is at hand: put $x_j = 0$ for j non-basic and the basic variable x_{B_i} corresponding to the i^{th} row equal to $\bar{b}_i$. If $\bar{b} \not\geqq 0$, then let $\bar{b}_r$ be the most negative component (actually, <u>any</u> negative component will do) and test to make sure that at least one component $\bar{a}_{rj}$ of the matrix $A_B^{-1} A$ is negative for some non-basic j (if none is negative, it can be shown that (P) is infeasible). Let k be defined so that

$$\frac{\bar{c}_k}{\bar{a}_{rk}} = \text{Maximum } \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : j \text{ non-basic and } \bar{a}_{rj} < 0 \right\},$$

and pivot on the element $\bar{a}_{rk}$ to obtain the detached coefficient array corresponding to the new set of basic variables $\{B - B_r + k\}$ (x_k is called the "entering," and x_{B_r} the "exiting" basic variable). If $\bar{c}_k > 0$ then $c_B A_B^{-1} b$ strictly decreases, and in any event the new $\bar{c}$ is also non-negative. The assumption of dual non-degeneracy means that $\bar{c}_j > 0$ for all non-basic j at each iteration, and can always be enforced by arbitrarily small perturbations of the problem data.

We are now in a position to make three key observations about the dual Simplex method.

<u>Lemma 2.1</u>: At any iteration of the dual Simplex method, the current basic solution is the unique optimal solution of (P_S) with S equal to the current set of non-basic variables ($S = M\text{-}B$).

<u>Proof</u>: The current basic solution is certainly feasible in (P_{M-B}). To show that it is optimal, by the Dual Theorem of linear programming it suffices to display a feasible solution to the dual of (P_{M-B}) with the same value of the objective function. One has only to verify, using $\bar{c} \geq 0$, that $(c_B A_B^{-1})$ is such a dual solution. Uniqueness of the optimal solution of (P_{M-B}) follows from the assumed non-degeneracy of the dual.

<u>Lemma 2.2</u>: If the dual Simplex method terminates because (P) is infeasible (i.e., if $\bar{b}_r < 0$ and $\bar{a}_{rj} \geq 0$ for all non-basic j at some tableau), then (P_S), with S equal to the current set of non-basic variables plus B_r, is infeasible.

Proof: By the Dual Theorem of linear programming, it is enough to show that the dual of (P_S) is feasible and has an unbounded optimum. It may be verified that $(c_B A_B^{-1}) + \theta(A_B^{-1})_{r.}$, where $(A_B^{-1})_{r.}$ is the r^{th} row of A_B^{-1} $(\bar{b}_r < 0)$, is feasible in the dual for all $\theta \geq 0$ and achieves an arbitrarily small value of the dual objective as $\theta \to \infty$.

Lemma 2.3: At any non-terminal iteration of the dual Simplex method, if x_k is the entering basic variable then $x_k > 0$ in the next basic solution.

Proof: The definition of the pivot operation implies that $x_k = (\bar{b}_r/\bar{a}_{rk})$ in the next basic solution. By selection, $\bar{b}_r < 0$ and $\bar{a}_{rk} < 0$.

Proof of Th. 2.1: The proof proceeds by induction on ν. At $\nu = 1$, S^o has been taken as $M\text{-}B^o$, the initial set of non-basic variables. Lemma 2.1 asserts that (P_{S^o}) has a unique solution x^{S^o}. Hence x^{S^o} must be the initial basic solution. Since $x_j^{S^o} = 0$ by definition for all non-basic j, $E_{S^o} = S^o$. Thus the assertion is true for $\nu = 1$.

Assume that the assertion is true for the ν_o^{th} iteration of the dual Simplex method. Either the ν_o^{th} iteration is terminal because (P) has been solved, or is terminal because (P) has been found to be infeasible, or is not terminal. In the first case, the reduction procedure also terminates with an optimal solution of (P). In the second case, by Lemma 2.2 the next sub-problem encountered by the reduction procedure is infeasible and therefore terminal. Consider now the third case. We shall show that the assertion of the theorem holds at the next iteration by detailing the operation of the reduction procedure starting at Step 2 of the current iteration.

Dual non-degeneracy implies that $f(x^S)$ decreases strictly at each iteration. Hence the trial set to be used at the $(\nu_o+1)^{st}$ iteration of the reduction procedure is $E_S \cup B_r$, where x_{B_r} is the most negative component of the current x^S. It follows from Lemmas 2.1 and 2.3 that $(P_{E_S \cup B_r})$ has a unique solution, and that all components indexed by $E_S \cup B_r$ vanish in this solution except for x_k, which is strictly positive. The assertion of the theorem now follows immediately.

It would not be at all surprising if a similar relation to van de Panne and Whinston's dual method for quadratic programming [9] could be established, in view of its close relation to the dual Simplex method.

III. DISCUSSION

There are two especially important types of difficulties that may arise in connection with certain potential applications of the reduction procedure. One is that there may be an infinite (possibly uncountable) number of constraints associated with (P). In this case most of the constraints can be expected to be redundant, and the number of constraints in the subproblems will usually not become unbounded. It may or may not be difficult to test for $V_S = \emptyset$ at Step 1, and a convergence proof must be constructed along different lines than the one given above. The other difficulty is that some or all of the constraints may only be implicitly available, or available only at high computational cost. This, of course, may make it difficult to identify <u>any</u> violated constraint, let alone an advantageous subset of V_S.

One or both of these difficulties are encountered by the so-called "cutting-plane" methods of mathematical programming, as typified by Gomory's method for integer linear programming [5] and Kelley's method for concave programming [7]. The term "plane" rather than "surface" has come into general use because such methods usually employ linear rather than nonlinear constraints.

Gomory's method can be viewed as the dual Simplex method applied to a linear program that is equivalent to the original integer program; equivalent in the sense that its feasible region coincides with the convex hull of the feasible integer solutions of the original program. Although there is a vast number of constraints, and most of them are not explicitly available, Gomory has shown how to generate an appropriate violated constraint

inexpensively if $V_S \neq \phi$ (in our terminology). This enables Step 2 to be carried out, and of course there is no difficulty in testing $V_S = \phi$. Convergence is finite, although "finite" is sometimes too large for practical purposes.

Kelley's method (see also Cheney and Goldstein [1], and the references therein) can be cast in the form of the present reduction procedure as follows. Define $G(x) \equiv \underset{i}{\text{Min}} \{g_i(x)\}$. ($G(x)$ is concave on X.) Then (P) can be replaced by

$$\text{Maximize}_{x \in X} \ f(x) \ \text{ subject to } \ G(x) \geq 0.$$

Kelley imposes sufficient conditions for the existence of a support $G(x') + \gamma_{x'} \cdot (x-x')$ to the graph of G for each x' in X, where $\gamma_{x'}$ is a fixed n-vector for each $x' \in X$. For fixed x', the support function is linear and has the defining property: $G(x') + \gamma_{x'} \cdot (x-x') \geq G(x)$ for all $x \in X$, with equality holding for $x = x'$. (If G is differentiable at x', $\gamma_{x'}$ is just the gradient of G at x'.) The family of support functions for G is introduced to permit the feasible region of (P) to be represented by the intersection of the half-spaces containing it; that is, to enable (P) to be replaced by

$$\underset{x \in X}{\text{Maximize}} \ f(x) \ \text{ subject to } \ G(x') + \gamma_{x'} \cdot (x-x') \geq 0, \quad \text{all } x' \in X',$$

where $X' = \{x' \in X : G(x') \leq 0\}$. It is easy to verify that this problem, which in addition to X is restricted by an infinite collection of linear constraints indexed by X', has the same feasible region as (P). Now apply the reduction procedure. Testing for $V_S = \phi$ is equivalent, of course, to testing $G(x^S) \geq 0$. (Since this test is unlikely ever to be satisfied

for truly nonlinear problems, in practice one would probably test $G(x^S) \geq -\epsilon$, where ϵ is some suitably small positive number.) If $S^0 = \emptyset$ and v is chosen to correspond to the constraint

$$G(x^S) + \gamma_{x^S}(x-x^S) \geq 0$$

at Step 2, then the result is Kelley's method with an explicit rule for dropping unneeded constraints (cf. [7,p. 710]). Note that the choice of v is not necessarily the constraint most violated at the current trial solution; but it <u>is</u> violated, and it would be computationally expensive to find one that is more so. Of course, since M ($\equiv X'$ here) is infinite our finite termination proof no longer holds. But Kelley was able to obtain a simple proof of convergence in the limit (cf. [1]) under rather mild conditions, and this suggests that convergence of the reduction procedure in the appropriate sense can also be shown for other interesting classes of problems with an infinite number of constraints.

REFERENCES

1. Cheney, E. W. and A. A. Goldstein, "Newton's Method for Convex Programming and Tchebycheff Approximations," Numerische Mathematik, 1, 253-268 (1959).

2. Dantzig, G. B., "Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming," Econometrica, 23, 174-183 (1955).

3. Geoffrion, A. M., "Reducing Concave Programs with Some Linear Constraints," forthcoming in SIAM Journal on Applied Mathematics.

4. Geoffrion, A. M., "A Markovian Procedure for Concave Programming with Some Linear Constraints," forthcoming in the Proceedings of the Fourth International Conference on Operational Research, Cambridge, Mass., August 29 - September 2, 1966.

5. Gomory, R. E., "An Algorithm for Integer Solutions to Linear Programs," appearing in Recent Advances in Mathematical Programming, R. L. Graves and P. Wolfe (Eds.), McGraw-Hill, New York, 1963.

6. Hadley, G., Linear Programming, Addison-Wesley, Reading, Mass., 1962.

7. Kelley, J. E., Jr., "The Cutting-Plane Method for Solving Convex Programs," SIAM Journal on Applied Mathematics, 8, 703-712 (1960).

8. Lemke, C. E., "The Dual Method of Solving the Linear Programming Problem," Naval Research Logistics Quarterly, 1, 36-47 (1954).

9. van de Panne, C. and A. Whinston, "The Simplex and the Dual Method for Quadratic Programming," Operations Research Quarterly, 15, 355-388 (1964).

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) Western Management Science Institute University of California at Los Angeles Los Angeles, California 90024	2a REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP

3 REPORT TITLE

Direct Reduction of Large Concave Programs

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)

Working Paper

5 AUTHOR(S) (Last name, first name, initial)

Arthur M. Geoffrion

6 REPORT DATE December, 1966	7a TOTAL NO. OF PAGES 16	7b NO. OF REFS 9

8a CONTRACT OR GRANT NO. Nonr 233(75) b. PROJECT NO. c. d.	9a ORIGINATOR'S REPORT NUMBER(S) Working Paper No. 111 9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10 AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited

11 SUPPLEMENTARY NOTES	12 SPONSORING MILITARY ACTIVITY

13 ABSTRACT

A very simple and flexible procedure is given for reducing a concave mathematical program to a finite sequence of subproblems, each involving a subset of the original constraints. While very much in the spirit of the "secondary constraint" idea, constraints are dropped as well as added from subproblem to subproblem. The reduction procedure is of quite general applicability, and is designed to be used with various concave programming algorithms for solving the subproblems. It seems particularly suited to facilitating the solution of problems with more constraints than could otherwise be handled. It is shown that an obvious specialization to linear programs yields the dual Simplex method, and that various "cutting-plane" methods can also be naturally described from the general viewpoint offered here.

Security Classification

14 KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Secondary constraints Mathematical programming Cutting-plane methods Dual Simplex methods						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

Security Classification